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On the Semiproportional Character Conjecture in Groups $\mathrm{Sp}_4(q)$

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Abstract—Previously, the author made the following conjecture: if a finite group has two semiproportional irreducible characters φ and ψ, then $\varphi(1) = \psi(1)$. In the present paper, a new confirmation of the conjecture is obtained. Namely, the conjecture is verified for symplectic groups $\mathrm{Sp}_4(q)$ and $\mathrm{PSp}_4(q)$.
Keywords: finite symplectic groups, character table, semiproportional characters, small interactions.

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INTRODUCTION

Irreducible characters φ and ψ of a finite group G are called *semiproportional* if they are not proportional and, for some normal subset D from G, the restrictions of φ and ψ to D and their restrictions to $G \setminus D$ are proportional. Earlier, the author obtained a complete description of all pairs of semiproportional irreducible characters (in other terms, of small D-blocks [1]) in sporadic simple groups [2]; in groups $\mathrm{L}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PGL}_2(q)$, and $\mathrm{GL}_2(q)$ [3]; in groups $\mathrm{PGL}_3(q)$, $\mathrm{GL}_3(q)$, $\mathrm{PGU}_3(q)$, and $\mathrm{GU}_3(q)$ [4]; in groups $\mathrm{L}_3(q)$, $\mathrm{SL}_3(q)$, $\mathrm{U}_3(q)$, and $\mathrm{SU}_3(q)$ [5]; and in groups $\mathrm{Sp}_4(q)$ for even q [6]. The following conjecture was proposed in [3].

Conjecture (on semiproportional characters). *If a finite group G has two semiproportional irreducible characters φ and ψ, then $\varphi(1) = \psi(1)$.*

The conjecture was confirmed for all the groups listed above. Some properties of an arbitrary group with a pair of semiproportional irreducible characters were studied in [7]. In [6], it was proved that simple groups $\mathrm{Sp}_4(q)$ for even q have no pairs of semiproportional irreducible characters and, in particular, the conjecture is true for these groups. Now, we will prove the conjecture for groups $\mathrm{Sp}_4(q)$ for odd q.

Theorem. *Let $G = \mathrm{Sp}_4(q)$, where q is odd. Then,*

(1) *The conjecture is true for the group G.*

(2) *If φ and ψ are semiproportional irreducible characters of the group G, then, in terms of its character table (see Section 2), $\{\varphi, \psi\}$ is contained either in X_i for $i \in \{1, \ldots, 13\}$ or in one of the sets Ξ_s and Ξ'_s for $s \in \{1, 3, 21, 22, 41, 42\}$.*

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This theorem and [6] immediately imply the following statement.

Corollary. The conjecture is true for groups $\mathrm{Sp}_4(q)$ and $\mathrm{PSp}_4(q)$ for any q.

Statement (2) of the theorem will be essentially used for describing pairs of semiproportional irreducible characters of groups $\mathrm{Sp}_4(q)$ and $\mathrm{PSp}_4(q)$ for odd q; we intend to give this description in the next paper.

We prove the theorem by analyzing character tables of the groups under consideration, which were obtained by Srinivasan [8]; we take into account the corrections made by Przygocki [9]. These tables are given in Section 2 in a more convenient form.

Notation. $\mathrm{Cl}\,(G)$ is the set of all classes of conjugate elements of the group G, g^G is the class of conjugate elements of the group G that contains the element g from G, classes of conjugate elements of a group will be called simply *classes of the group*, $\mathrm{Irr}(G)$ is the set of all irreducible characters of G, $\dot{\cup}$ is a sign for the union of pairwise nonintersecting sets, $\mathbb{Z}$ is the set of all integers, $\widehat{Z}$ is the set of all algebraic integers, and $A := B$ (read: A is B by definition) means that A denotes B.

1. PRELIMINARY RESULTS

Proposition 1.1 (follows from [1, Theorem 8Z3]). *Let φ and ψ be semiproportional irreducible characters of a group G, and let $g_1, g_2 \in G$. Then,*

(1) $\varphi(g) = 0 \iff \psi(g) = 0$ *for all* $g \in G$;

(2) *if* $\varphi(g_1)$ *and* $\varphi(g_2)$ *are not zeros, then* $\dfrac{\varphi(g_i)}{\psi(g_i)} \in \left\{\dfrac{\varphi(1)}{\psi(1)}, -\dfrac{\psi(1)}{\varphi(1)}\right\}$ $(i \in \{1,2\})$ *and, in particular, either* $\dfrac{\varphi(g_1)}{\psi(g_1)} = \dfrac{\varphi(g_2)}{\psi(g_2)}$ *or* $\dfrac{\varphi(g_1)}{\psi(g_1)}\dfrac{\varphi(g_2)}{\psi(g_2)} = -1$;

(3) *if* $|\varphi(g_1)| = |\psi(g_1)| \neq 0$, *then* $\varphi(g) = \pm\,\psi(g)$ *for all* $g \in G$ (*in particular,* $\varphi(1) = \psi(1)$).

Proposition 1.2 [7, Corollary of Lemma 2.1]. *If φ and ψ are semiproportional irreducible characters of a group G and G has an element g such that $\varphi(g)$ or $\psi(g)$ is invertible in $\widehat{\mathbb{Z}}$ (for example, if $|\varphi(g)| = 1$ or $|\psi(g)| = 1$), then one of the numbers $\varphi(1)$ and $\psi(1)$ divides the other.*

2. CLASSES AND CHARACTERS OF GROUPS $\mathrm{Sp}_4(q)$ WITH ODD q

Let $G = \mathrm{Sp}_4(q)$, where q is odd. Then, G is a group of order $q^4(q^4-1)(q^2-1)$, $Z(G) = \langle z\rangle$ is a group of order 2, and $G/Z(G) = \mathrm{PSp}_4(q)$ is a simple group. By definition, $G = \{g \in \mathrm{GL}_4(q) \mid {}^t gAg = A\}$, where ${}^t g$ is the transpose of the matrix g and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The character table $\mathrm{X}(G)$ of the group G was obtained by Srinivasan in [8]. It is given below in Tables A, B1, B2, C, D, A′, B1′, B2′, C′, and D′ with the corrections made by Przygocki [9]. The table $\mathrm{X}(G)$ contains two auxiliary rows and, then, 32 indexed rows, which consist of values of several characters specified in the second column of the table. Similarly, after two auxiliary columns, there are 23 indexed columns, which consist of the values of irreducible characters on classes of conjugate elements whose representatives are written in the second row.

Elements of the character table are sums of roots of unity $\zeta_n := e^{\frac{2\pi \mathbf{i}}{n}}$ ($\mathbf{i}$ is the imaginary unit) for some natural n. The roots either are integers or are expressed in terms of the following numbers:

$$\sigma := \zeta_{q^2-1}\,, \qquad \alpha_m := \zeta_{q-1}^m + \zeta_{q-1}^{-m} = 2\cos\frac{2m\pi}{q-1}\,,$$

$$\tau := \zeta_{q^2+1}\,, \qquad \beta_m := \zeta_{q+1}^m + \zeta_{q+1}^{-m} = 2\cos\frac{2m\pi}{q+1}\,,$$

$$t := (q-1)/2\,, \quad \varepsilon := (-1)^t\,, \quad b := (-1-\varepsilon\sqrt{\varepsilon q})/2\,, \quad b^* := (-1+\varepsilon\sqrt{\varepsilon q})/2\,,$$

$$s(i,j) := (-1)^i + (-1)^j\,, \quad d(i,j) := (-1)^i - (-1)^j\,.$$

For a more compact presentation of the tables, we write

$$q_- := q-1 \quad \text{and} \quad q_+ := q+1.$$

Note that $\alpha_m = 0 \Leftrightarrow m = ((q-1)/4)(2k+1)$ and $\beta_m = 0 \Leftrightarrow m = ((q+1)/4)(2k+1)$ $(k \in \mathbb{Z})$.

Classes of conjugate elements and irreducible characters of the group G depend on several parameters, which are contained in the sets

$$T_1 := \{1,2,\ldots,(q-3)/2\}, \quad T_2 := \{1,2,\ldots,(q-1)/2\}, \quad R_1 := \{1,2,\ldots,(q^2-1)/4\},$$

and R_2, where the latter set consists of $(q-1)^2/4$ positive integers i (starting from with 1) such that the numbers σ^i, σ^{-i}, σ^{qi}, and σ^{-qi} are pairwise different.

2.1. Classes of conjugate elements of the group *G*. We have $\mathrm{Cl}\,(G) = A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, D$, where the sets A, B, C, and D are described below.

For any element g of the group G, denote by g' the element gz in the case when $(gz)^G \neq g^G$. It is clear that $C_G(g') = C_G(g)$. The classes g^G and h^G of the group G $(g,h \in G)$ are called *algebraically conjugate* if there exists a positive integer m such that $m \le |G|$, $(m,|G|) = 1$, and $h^G = (g^m)^G$ (then, by [1, 2A14 and 2A15], for any $\chi \in \mathrm{Irr}(G)$, $\chi(h) = \chi(g)^\alpha$, where α is an automorphism of the field $Q(\zeta_{|G|})$ that takes $\zeta_{|G|}$ to $(\zeta_{|G|})^m$).

The set A consists of 14 classes of conjugate elements with representatives $a_1 = 1$, $a_1' = z$, a_{21}, a_{21}', a_{22}, a_{22}', a_{31}, a_{31}', a_{32}, a_{32}', a_{41}, a_{41}', a_{42}, and a_{42}'. Among these classes, there are exactly four (unordered) pairs of algebraically conjugate classes:

$$\{a_{21}{}^G, a_{22}{}^G\}, \;\; \{a_{21}'{}^G, a_{22}'{}^G\}, \;\; \{a_{41}{}^G, a_{42}{}^G\}, \;\; \{a_{41}'{}^G, a_{42}'{}^G\}. \tag{2.1}$$

Here, $|C_G(a_{21})| = |C_G(a_{22})| = 2q^4(q^2-1)$, $|C_G(a_{31})| = 2q^3(q-1)$, $|C_G(a_{32})| = 2q^3(q+1)$, and $|C_G(a_{41})| = |C_G(a_{42})| = 2q^2$.

The set B is defined by the formula $B = \dot\cup_{i=1}^{9} B_i$, where

$B_1 = \{\, b_1(i)^G \mid i \in R_1\}$ $\;(\,|B_1| = |R_1| = (q^2-1)/4,\;\; |C_G(b_1(i))| = q^2+1\,)$,

$B_2 = \{\, b_2(i)^G \mid i \in R_2\}$ $\;(\,|B_2| = |R_2| = (q-1)^2/4,\;\; |C_G(b_2(i))| = q^2-1\,)$,

$B_3 = \{\, b_3(i,j)^G \mid i,j \in T_1, i<j\}$
$\quad(\,|B_3| = |\{(i,j) \in T_1 \times T_1 \mid i<j\}| = (q-3)(q-5)/8,\;\; |C_G(b_3(i,j))| = (q-1)^2\,)$,

$B_4 = \{\, b_4(i,j)^G \mid i,j \in T_2, i<j\}$
$\quad(\,|B_4| = |\{(i,j) \in T_2 \times T_2 \mid i<j\}| = (q-1)(q-3)/8,\;\; |C_G(b_4(i,j))| = (q+1)^2\,)$,

$B_5 = \{\, b_5(i,j)^G \mid i \in T_2, j \in T_1\}$ $\;(\,|B_5| = |T_2 \times T_1| = (q-1)(q-3)/4,\;\; |C_G(b_5(i,j))| = q^2-1\,)$,

$B_6 = \{\, b_6(i)^G \mid i \in T_2\}\ \ (\,|B_6| = |T_2| = (q-1)/2,\ \ |C_G(b_6(i))| = q(q+1)(q^2-1)\,),$
$B_7 = \{\, b_7(i)^G \mid i \in T_2\}\ \ (\,|B_7| = |T_2| = (q-1)/2,\ \ |C_G(b_7(i))| = q(q+1)\,),$
$B_8 = \{\, b_8(i)^G \mid i \in T_1\}\ \ (\,|B_8| = |T_1| = (q-3)/2,\ \ |C_G(b_8(i))| = q(q-1)(q^2-1)\,),$
$B_9 = \{\, b_9(i)^G \mid i \in T_1\}\ \ (\,|B_9| = |T_1| = (q-3)/2,\ \ |C_G(b_9(i))| = q(q-1)\,).$

Further, $C = C_1 \,\dot\cup\, C'_1 \,\dot\cup\, C_{21} \,\dot\cup\, C'_{21} \,\dot\cup\, C_{22} \,\dot\cup\, C'_{22} \,\dot\cup\, C_3 \,\dot\cup\, C'_3 \,\dot\cup\, C_{41} \,\dot\cup\, C'_{41} \,\dot\cup\, C_{42} \,\dot\cup\, C'_{42}$, where

$C_1 = \{\, c_1(i)^G \mid i \in T_2\},\ \ C'_1 = \{\, (c_1(i)')^G \mid i \in T_2\}$
$(\,|C_1| = |C'_1| = |T_2| = (q-1)/2,\ \ |C_G(c_1(i))| = |C_G(c_1(i)')| = q(q+1)(q^2-1)\,),$

$C_{21} = \{\, c_{21}(i)^G \mid i \in T_2\}\,,\ C'_{21} = \{\, (c_{21}(i)')^G \mid i \in T_2\}$
$(\,|C_{21}| = |C'_{21}| = |T_2| = (q-1)/2,\ \ |C_G(c_{21}(i))| = |C_G(c_{21}(i)')| = 2q(q+1)\,),$

$C_{22} = \{\, c_{22}(i)^G \mid i \in T_2\}\,,\ \ C'_{22} = \{\, (c_{22}(i)')^G \mid i \in T_2\}$
$(\,|C_{22}| = |C'_{22}| = |T_2| = (q-1)/2,\ \ |C_G(c_{22}(i))| = |C_G(c_{22}(i)')| = 2q(q+1)\,),$

$C_3 = \{\, c_3(i)^G \mid i \in T_1\}\,,\ \ C'_3 = \{\, (c_3(i)')^G \mid i \in T_1\}$
$(\,|C_3)| = |C'_3)| = |T_1| = (q-1)/2,\ \ |C_G(c_3(i))| = |C_G(c_3(i)')| = q(q-1)(q^2-1)\,),$

$C_{41} = \{\, c_{41}(i)^G \mid i \in T_1\}\,,\ C'_{41} = \{\, (c_{41}(i)')^G \mid i \in T_1\}$
$(\,|C_{41}| = |C'_{41}| = |T_1|/2 = (q-3)/2,\ \ |C_G(c_{41}(i))| = |C_G(c_{41}(i)')| = 2q(q-1)\,),$

$C_{42} = \{\, c_{42}(i)^G \mid i \in T_1\}\,,\ \ C'_{42} = \{\, (c_{42}(i)')^G \mid i \in T_1\}$
$(\,|C_{42}| = |C'_{42}| = |T_1|/2 = (q-3)/2,\ \ |C_G(c_{42}(i))| = |C_G(c_{42}(i)')| = 2q(q-1)\,).$

The set D consists of nine classes of conjugate elements with representatives d_1, d_{21}, d'_{21}, d_{22}, d'_{22}, d_{24}, d_{31}, d_{32}, d_{33}, and d_{34}. Among these classes, there are exactly four pairs of algebraically conjugate classes:

$$\{{d_{21}}^G, {d_{22}}^G\},\ \ \{{d'_{21}}^G, {d'_{22}}^G\},\ \ \{{d_{31}}^G, {d_{33}}^G\},\ \ \{{d_{32}}^G, {d_{34}}^G\}. \tag{2.2}$$

Here, $|C_G(d_1)| = q^2(q^2-1)^2$, $|C_G(d_{21})| = |C_G(d_{22})| = 2q^2(q^2-1)$, and $|C_G(d_{31})| = |C_G(d_{32})| = |C_G(d_{33})| = |C_G(d_{34})| = 4q^2$.

Remark 2.1. Representatives of algebraically conjugate classes of conjugate elements are written in the same column, and this column contains the values of irreducible characters for only one of them; the values of irreducible characters for the other class are obtained from their values for the first class by changing b for b^* and b^* for b (the values b and b^* are given below).

Remark 2.2. Elements g and g' are written in the same column, which contains only the values of characters $\chi \in \mathrm{Irr}(G)$ on g. To obtain the value $\chi(g')$, one should take the entry in the second column of the same row and set in it $D = \chi(g)$. (In the second column, we have $D = \chi(a_1)$.)

2.2. Irreducible characters of the group *G*. $\mathrm{Irr}(G) = \Theta \,\dot\cup\, \mathrm{X} \,\dot\cup\, \Xi \,\dot\cup\, \Xi' \,\dot\cup\, \Phi$, where

$\Theta := \{\theta_0, \theta_1, \ldots, \theta_{13}\}$, $\theta_0 = 1_G$;

$\Phi := \{\varphi_1, \ldots, \varphi_9\}$;

$\mathrm{X} := \dot\bigcup_{i=1}^{9} \mathrm{X}_i$, where X_i consists of characters of the form $\chi_i^{(k,l)}$ (for $i \in \{3,4,5\}$) or of the form $\chi_i^{(k)}$ for certain k and l, which are specified in the character table;

$\Xi := \dot\bigcup_{s\in S} \Xi_s$ and $\Xi' := \dot\bigcup_{s\in S} \Xi'_s$, where $S := \{1,3,21,22,41,42\}$ and the sets Ξ_s and Ξ'_s consist of characters $\xi_s^{(k)}$ and $\xi_s'^{(k)}$, respectively, for the values k specified in the character table.

It is easy to see that

$|\Theta| = |A|$;

$|\mathrm{X}_i| = |B_i|$ for $i \in \{1, 2, \ldots, 9\}$ and, consequently, $|\mathrm{X}| = |B|$;
$|\Xi_i| = |\Xi'_i| = |C_i| = |C'_i|$ for $i \in \{1, 3, 21, 22, 41, 42\}$ and, consequently, $|\Xi| = |\Xi'| = |C| = |C'|$;
$|\Phi| = |D|$.

Among the characters $\theta_1, \ldots, \theta_{13}, \varphi_1, \ldots, \varphi_9$, there are exactly eight pairs of algebraically conjugate characters:

$$\{\theta_1, \theta_2\},\ \{\theta_3, \theta_4\},\ \{\theta_5, \theta_6\},\ \{\theta_7, \theta_8\},\ \{\varphi_1, \varphi_2\},\ \{\varphi_3, \varphi_4\},\ \{\varphi_5, \varphi_6\},\ \{\varphi_7, \varphi_8\}. \tag{2.3}$$

Among the characters from $\Xi \cup \Xi'$, there are exactly the following pairs of algebraically conjugate characters:

$$\{\xi_{21}^{(k)}, \xi_{22}^{(k)}\},\quad \{\xi_{21}^{\prime(k)}, \xi_{22}^{\prime(k)}\},\quad \{\xi_{41}^{(k)}, \xi_{42}^{(k)}\},\quad \{\xi_{41}^{\prime(k)}, \xi_{42}^{\prime(k)}\}\ \text{ for corresponding } k. \tag{2.4}$$

Remark 2.3. In the character table of the group G, algebraically conjugate characters are given in the same row, which contains the value of only one character from each pair. The value of the second character on any element g is obtained from the value of the first character on g by changing b for b^* and b^* for b.

3. PROOF OF THE THEOREM

Let us consider the character table $\mathrm{X}(G)$ of the group G (see Tables A–D$'$) using Propositions 1.1 and 1.2. The fact that (by statement (1) of Proposition 1.1) semiproportional irreducible characters of the group have the same set of roots (zeros) will be used further without special mention.

Let $\widetilde{i}$ and $\widehat{j}$ denote the row and column of the character table $\mathrm{X}(G)$ marked by the indices i and j, respectively. If $g \in G$ and $\chi(g) = c$ for all characters χ from the row $\widetilde{i}$, then we write $\widetilde{i}(g) = c$. If the intersection of the row $\widetilde{i}$ and column $\widehat{j}$ contains a specific number without parameters and numbers b or b^*, then we call this number the *value of* $\widetilde{i}$ *on* $\widehat{j}$ and denote it by $\widetilde{i}(\widehat{j})$.

Rows $\widetilde{i}$ and $\widetilde{j}$ will be called *semiproportional* if some character from the row $\widetilde{i}$ is semiproportional to some character from the row $\widetilde{j}$.

We will say that rows $\widetilde{i}$ and $\widetilde{j}$ do not have equal roots on an element $g \in G$ if exactly one of the values $\widetilde{i}(g)$ and $\widetilde{j}(g)$ is zero. By Statement (1) of Proposition 1.1, rows that do not have equal roots on at least one element of the group are not semiproportional.

Assume that ψ and φ are semiproportional irreducible characters of the group G lying in rows with indices m and n, respectively (consequently, $\widetilde{m}$ is semiproportional to $\widetilde{n}$). Contrary to statement (2) of the theorem, let $m \neq n$.

First of all, note that $\{m, n\} \neq \{25, 26\}$, applying statement (2) of Proposition 1.1 for $g_1 = a_1$ and $g_2 = a_{31}$. Further, as seen from the tables, for any row $\widetilde{s}$ different from $\widetilde{25}$ and $\widetilde{26}$, there exists an element g on which the characters of this row take the value ± 1. Therefore, according to Proposition 1.2,

$$\text{one of the numbers } \varphi(1) \text{ and } \psi(1) \text{ divides the other.} \tag{3.1}$$

If $\varphi(1) = \psi(1)$, then, as seen from the tables, the pair $\{m, n\}$ is one of the pairs $\{8, 9\}$, $\{12, 15\}$, $\{16, 20\}$, $\{17, 21\}$, $\{18, 22\}$, $\{19, 23\}$, and $\{24, 27\}$. Then, according to statement (3) of Proposition 1.1, in each of these cases, $\varphi(g) = \psi(g)$ for all $g \in G$. However, this equality is contradictory: for $g = a_{31}$ in the first six cases and for $g = a_{41}$ in the last case. Consequently, $\psi(1) \neq \varphi(1)$; then, by statement (3) of Proposition 1.1, a stronger assertion is true:

$$\text{there is no element } g \text{ in } G \text{ such that } |\varphi(g)| = |\psi(g)| \neq 0. \tag{3.2}$$

Table A. Character table of the group $\mathrm{Sp}_4(q)$, q is odd

		1	2	3	4	5	6
		a_1	a'_1	a_{21}, a'_{21}, a_{22}, a'_{22}	a_{31}, a'_{31}	a_{32}, a'_{32}	a_{41}, a'_{41}, a_{42}, a'_{42}
1	θ_0	1	D	1	1	1	1
2	θ_1, θ_2	$q^2(q^2+1)/2$	D	$-q^2b$	0	0	0
3	θ_3, θ_4	$(q^2+1)/2$	D	$q_+/2+qb$	$q_+/2$	$-q_-/2$	$-b^*$
4	θ_5, θ_6	$q^2(q^2-1)/2$	$-D$	q^2b	0	0	0
5	θ_7, θ_8	$(q^2-1)/2$	$-D$	$q_-/2+qb$	$q_-/2$	$-q_+/2$	b
6	θ_9	$q{q_+}^2/2$	D	$qq_+/2$	q	0	0
7	θ_{10}	$q{q_-}^2/2$	D	$-qq_-/2$	0	q	0
8	θ_{11}	$q(q^2+1)/2$	D	$-qq_-/2$	q	0	0
9	θ_{12}	$q(q^2+1)/2$	D	$qq_+/2$	0	q	0
10	θ_{13}	q^4	D	0	0	0	0
11	$\chi_1^{(k)}$ $(k \in R_1)$	$(q^2-1)^2$	$(-1)^kD$	$-(q^2-1)$	$-q_-$	q_+	1
12	$\chi_2^{(k)}$ $(k \in R_2)$	q^4-1	$(-1)^kD$	q^2-1	$-q_+$	q_-	1
13	$\chi_3^{(k,l)}$ $(k,l \in T_1, k<l)$	${q_+}^2(q^2+1)$	$(-1)^{k+l}D$	${q_+}^2$	$3q+1$	q_+	1
14	$\chi_4^{(k,l)}$ $(k,l \in T_2, k<l)$	${q_-}^2(q^2+1)$	$(-1)^{k+l}D$	${q_-}^2$	$-q_-$	$1-3q$	1
15	$\chi_5^{(k,l)}$ $(k \in T_2, l \in T_1)$	q^4-1	$(-1)^{k+l}D$	$-(q^2+1)$	q_-	$-q_+$	-1
16	$\chi_6^{(k)}$ $(k \in T_2)$	$q_-(q^2+1)$	D	q_-	-1	$2q-1$	-1
17	$\chi_7^{(k)}$ $(k \in T_2)$	$qq_-(q^2+1)$	D	qq_-	$-q$	$-q$	0

Table B1. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		7	8	9	10
		$b_1(i)$ $(i \in R_1)$	$b_2(i)$ $(i \in R_2)$	$b_3(i,j)$ $(i,j \in T_1, i < j)$	$b_4(i,j)$ $(i,j \in T_2, i < j)$
1	θ_0	1	1	1	1
2	θ_1, θ_2	0	$(-1)^{i+1}$	$(-1)^{i+j}$	$(-1)^{i+j}$
3	θ_3, θ_4	0	$(-1)^{i}$	$(-1)^{i+j}$	$(-1)^{i+j}$
4	θ_5, θ_6	$(-1)^{i}$	0	0	0
5	θ_7, θ_8	$(-1)^{i+1}$	0	0	0
6	θ_9	-1	0	2	0
7	θ_{10}	1	0	0	-2
8	θ_{11}	0	1	1	-1
9	θ_{12}	0	-1	1	-1
10	θ_{13}	1	-1	1	1
11	$\chi_1^{(k)}$	$\tau^{ik} + \tau^{-ik} +$ $+\tau^{qik} + \tau^{-qik}$	0	0	0
12	$\chi_2^{(k)}$	0	$-(\sigma^{ik} + \sigma^{-ik} +$ $+\sigma^{qik} + \sigma^{-qik})$	0	0
13	$\chi_3^{(k,l)}$	0	0	$\alpha_{ik}\alpha_{jl} + \alpha_{il}\alpha_{jk}$	0
14	$\chi_4^{(k,l)}$	0	0	0	$\beta_{ik}\beta_{jl} + \beta_{il}\beta_{jk}$
15	$\chi_5^{(k,l)}$	0	0	0	0
16	$\chi_6^{(k)}$	0	$-\beta_{ik}$	0	$-\beta_{ik}\beta_{jk}$
17	$\chi_7^{(k)}$	0	$-\beta_{ik}$	0	$\beta_{ik}\beta_{jk}$

Table B2. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		11	12	13	14	15
		$b_5(i,j)$ $(i \in T_2, j \in T_1)$	$b_6(i)$ $(i \in T_2)$	$b_7(i)$ $(i \in T_2)$	$b_8(i)$ $(i \in T_1)$	$b_9(i)$ $(i \in T_1)$
1	θ_0	1	1	1	1	1
2	θ_1, θ_2	0	$-q$	0	q	0
3	θ_3, θ_4	0	1	1	1	1
4	θ_5, θ_6	$(-1)^{i+j+1}$	0	0	0	0
5	θ_7, θ_8	$(-1)^{i+j+1}$	0	0	0	0
6	θ_9	0	0	0	q_+	1
7	θ_{10}	0	q_-	-1	0	0
8	θ_{11}	-1	q	0	1	1
9	θ_{12}	1	-1	-1	q	0
10	θ_{13}	-1	$-q$	0	q	0
11	$\chi_1^{(k)}$	0	0	0	0	0
12	$\chi_2^{(k)}$	0	$-q_+\beta_{ik}$	$-\beta_{ik}$	$q_-\alpha_{ik}$	$-\alpha_{ik}$
13	$\chi_3^{(k,l)}$	0	0	0	$q_+\alpha_{ik}\alpha_{il}$	$\alpha_{ik}\alpha_{il}$
14	$\chi_4^{(k,l)}$	0	$-q_-\beta_{ik}\beta_{il}$	$\beta_{ik}\beta_{il}$	0	0
15	$\chi_5^{(k,l)}$	$-\beta_{ik}\alpha_{jl}$	0	0	0	0
16	$\chi_6^{(k)}$	0	$-\beta_{2ik} + q_-$	$-\beta_{2ik} - 1$	q_-	-1
17	$\chi_7^{(k)}$	0	$-q\beta_{2ik} - q_-$	1	q_-	-1

Table C. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		16	17	18	19
		$c_1(i), c_1(i)'$ $(i \in T_2)$	$c_{21}(i), c_{21}(i)'$, $c_{22}(i), c_{22}(i)'$ $(i \in T_2)$	$c_3(i), c_3(i)'$ $(i \in T_1)$	$c_{41}(i), c_{41}(i)'$, $c_{42}(i), c_{42}(i)'$ $(i \in T_1)$
1	θ_0	1	1	1	1
2	θ_1, θ_2	$(-1)^{i+1}q_-/2$	$(-1)^{i+1}b$	$(-1)^i q_+/2$	$(-1)^{i+1}b$
3	θ_3, θ_4	$(-1)^{i+1}q_-/2$	$(-1)^{i+1}b$	$(-1)^i q_+/2$	$(-1)^{i+1}b^*$
4	θ_5, θ_6	$(-1)^{i+1}q_+/2$	$(-1)^i b$	$(-1)^i q_-/2$	$(-1)^i b$
5	θ_7, θ_8	$(-1)^{i+1}q_+/2$	$(-1)^i b^*$	$(-1)^i q_-/2$	$(-1)^i b$
6	θ_9	0	0	q_+	1
7	θ_{10}	q_-	-1	0	0
8	θ_{11}	-1	-1	q	0
9	θ_{12}	q	0	1	1
10	θ_{13}	$-q$	0	q	0
11	$\chi_1^{(k)}$	0	0	0	0
12	$\chi_2^{(k)}$	0	0	0	0
13	$\chi_3^{(k,l)}$	0	0	$q_+(\alpha_{ik} + \alpha_{il})$	$\alpha_{ik} + \alpha_{il}$
14	$\chi_4^{(k,l)}$	$-q_-(\beta_{ik} + \beta_{il})$	$\beta_{ik} + \beta_{il}$	0	0
15	$\chi_5^{(k,l)}$	$-q_+\beta_{ik}$	$-\beta_{ik}$	$q_-\alpha_{il}$	$-\alpha_{il}$
16	$\chi_6^{(k)}$	$q_-\beta_{ik}$	$-\beta_{ik}$	0	0
17	$\chi_7^{(k)}$	$-q_-\beta_{ik}$	β_{ik}	0	0

Table D. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		20	21	22	23
		d_1	d_{21}, d'_{21}, d_{22}, d'_{22}	d_{31}, d_{33}	d_{32}, d_{34}
1	θ_0	1	1	1	1
2	θ_1, θ_2	$(-1)^t q$	$-(-1)^t qb$	0	0
3	θ_3, θ_4	$(-1)^t q$	$(-1)^t(q_+ + 2b)/2$	$(-1)^t(b - b^*)$	0
4	θ_5, θ_6	0	$-(-1)^t qb$	0	0
5	θ_7, θ_8	0	$(-1)^t(q_- - 2b)/2$	0	$(-1)^t(b - b^*)$
6	θ_9	${q_+}^2/2$	$q_+/2$	$q_+/2$	$-q_-/2$
7	θ_{10}	$-{q_-}^2/2$	$q_-/2$	$q_-/2$	$-q_+/2$
8	θ_{11}	$(q^2-1)/2 + q$	$q_-/2$	$-q_-/2$	$q_-/2$
9	θ_{12}	$(1-q^2)/2 + q$	$q_+/2$	$-q_-/2$	$q_+/2$
10	θ_{13}	q^2	0	0	0
11	$\chi_1^{(k)}$ $(k \in R_1)$	0	0	0	0
12	$\chi_2^{(k)}$ $(k \in R_2)$	0	0	0	0
13	$\chi_3^{(k,l)}$ $(k < l,$ $k, l \in T_1)$	${q_+}^2 s(k,l)$	$q_+ s(k,l)$	$s(k,l)$	$s(k,l)$
14	$\chi_4^{(k,l)}$ $(k < l,$ $k, l \in T_2)$	${q_-}^2 s(k,l)$	$-q_- s(k,l)$	$s(k,l)$	$s(k,l)$
15	$\chi_5^{(k,l)}$ $(k \in T_2, l \in T_1)$	$(q^2-1)s(k,l)$	$-s(k,l) - qd(k,l)$	$-s(k,l)$	$-s(k,l)$
16	$\chi_6^{(k)}$ $(k \in T_2)$	$(-1)^{k+1}{q_-}^2$	$(-1)^k q_-$	$(-1)^{k+1}$	$(-1)^{k+1}$
17	$\chi_7^{(k)}$ $(k \in T_2)$	$(-1)^k {q_-}^2$	$(-1)^{k+1} q_-$	$(-1)^k$	$(-1)^k$

Table A′. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		1	2	3	4	5	6
		a_1	a_1'	a_{21}, a_{21}', a_{22}, a_{22}'	a_{31}, a_{31}'	a_{32}, a_{32}'	a_{41}, a_{41}', a_{42}, a_{42}'
18	$\chi_8^{(k)}$ $(k \in T_1)$	$q_+(q^2+1)$	D	q_+	$2q+1$	1	1
19	$\chi_9^{(k)}$ $(k \in T_1)$	$qq_+(q^2+1)$	D	qq_+	q	q	0
20	$\xi_1^{(k)}$ $(k \in T_2)$	$q_-(q^2+1)$	$(-1)^k D$	$-q^2+q_-$	q_-	q_-	-1
21	${\xi'}_1^{(k)}$ $(k \in T_2)$	$qq_-(q^2+1)$	$(-1)^k D$	$-q$	0	$-2q$	0
22	$\xi_3^{(k)}$ $(k \in T_1)$	$q_+(q^2+1)$	$(-1)^k D$	q^2+q_+	q_+	q_+	1
23	${\xi'}_3^{(k)}$ $(k \in T_1)$	$qq_+(q^2+1)$	$(-1)^k D$	q	$2q$	0	0
24	$\xi_{21}^{(k)}, \xi_{22}^{(k)}$ $(k \in T_2)$	$(q^4-1)/2$	$(-1)^{k+t} D$	$-q_+/2+qq_-b$	$q_-/2$	$-q_+/2$	b^*
25	${\xi'}_{21}^{(k)}, {\xi'}_{22}^{(k)}$ $(k \in T_2)$	${q_-}^2(q^2+1)/2$	$(-1)^{k+t+1} D$	$-q_-/2-qq_-b$	$-q_-/2$	$(1-3q)/2$	$-b^*$
26	$\xi_{41}^{(k)}, \xi_{42}^{(k)}$ $(k \in T_1)$	${q_+}^2(q^2+1)/2$	$(-1)^{k+t} D$	$q_+/2-qq_+b$	$(3q+1)/2$	$q_+/2$	$-b$
27	${\xi'}_{41}^{(k)}, {\xi'}_{42}^{(k)}$ $(k \in T_1)$	$(q^4-1)/2$	$(-1)^{k+t+1} D$	$q_-/2+qq_+b$	$q_-/2$	$-q_+/2$	b
28	φ_1, φ_2	$q_-(q^2+1)/2$	$(-1)^{t+1} D$	$-{q_-}^2/2+qb$	$q_-/2$	$q_-/2$	b
29	φ_3, φ_4	$qq_-(q^2+1)/2$	$(-1)^{t+1} D$	$qq_-/2+q^2b$	0	$-q$	0
30	φ_5, φ_6	$q_+(q^2+1)/2$	$(-1)^t D$	${q_+}^2/2+qb^*$	$q_+/2$	$q_+/2$	$-b$
31	φ_7, φ_8	$qq_+(q^2+1)/2$	$(-1)^t D$	$qq_+/2+q^2b^*$	q	0	0
32	φ_9	$q(q^2+1)$	D	q	q	q	0
	Number of classes	1	1	4	2	2	4

Table B1′. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		7	8	9	10
		$b_1(i)$ $(i \in R_1)$	$b_2(i)$ $(i \in R_2)$	$b_3(i,j)$ $(i,j \in T_1, i<j)$	$b_4(i,j)$ $(i,j \in T_2, i<j)$
18	$\chi_8^{(k)}$	0	α_{ik}	$\alpha_{ik}\alpha_{jk}$	0
19	$\chi_9^{(k)}$	0	$-\alpha_{ik}$	$\alpha_{ik}\alpha_{jk}$	0
20	$\xi_1^{(k)}$	0	0	0	$-\beta_{ik}-\beta_{jk}$
21	${\xi'}_1^{(k)}$	0	0	0	$\beta_{ik}+\beta_{jk}$
22	$\xi_3^{(k)}$	0	0	$\alpha_{ik}+\alpha_{jk}$	0
23	${\xi'}_3^{(k)}$	0	0	$\alpha_{ik}+\alpha_{jk}$	0
24	$\xi_{21}^{(k)}, \xi_{22}^{(k)}$	0	0	0	0
25	${\xi'}_{21}^{(k)}, {\xi'}_{22}^{(k)}$	0	0	0	$(-1)^j\beta_{ik}+(-1)^i\beta_{jk}$
26	$\xi_{41}^{(k)}, \xi_{42}^{(k)}$	0	0	$(-1)^j\alpha_{ik}+(-1)^i\alpha_{jk}$	0
27	${\xi'}_{41}^{(k)}, {\xi'}_{42}^{(k)}$	0	0	0	0
28	φ_1, φ_2	0	0	0	$-s(i,j)$
29	φ_3, φ_4	0	0	0	$s(i,j)$
30	φ_5, φ_6	0	0	$s(i,j)$	0
31	φ_7, φ_8	0	0	$-s(i,j)$	0
32	φ_9	0	0	$2(-1)^{i+j}$	$-2(-1)^{i+j}$
	Number of classes	$(q^2-1)/4$	$(q^2-1)/4$	$(q-3)(q-5)/4$	$(q-1)(q-3)/4$

Table B2′. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		11	12	13	14	15
		$b_5(i,j)$ $(i \in T_2, j \in T_1)$	$b_6(i)$ $(i \in T_2)$	$b_7(i)$ $(i \in T_2)$	$b_8(i)$ $(i \in T_1)$	$b_9(i)$ $(i) \in T_1)$
18	$\chi_8^{(k)}$	0	q_+	1	$\alpha_{2ik} + q_+$	$\alpha_{2ik} + 1$
19	$\chi_9^{(k)}$	0	$-q_+$	-1	$q\alpha_{2ik} + q_+$	1
20	$\xi_1^{(k)}$	$-\beta_{ik}$	$q_-\beta_{ik}$	$-\beta_{ik}$	0	0
21	${\xi'}_1^{(k)}$	$-\beta_{ik}$	$-q_-\beta_{ik}$	β_{ik}	0	0
22	$\xi_3^{(k)}$	α_{jk}	0	0	$q_+\alpha_{ik}$	α_{ik}
23	${\xi'}_3^{(k)}$	$-\alpha_{jk}$	0	0	$q_+\alpha_{ik}$	α_{ik}
24	$\xi_{21}^{(k)}, \xi_{22}^{(k)}$	$(-1)^{j+1}\beta_{ik}$	0	0	0	0
25	${\xi'}_{21}^{(k)}, {\xi'}_{22}^{(k)}$	0	$(-1)^{i+1}q_-\beta_{ik}$	$(-1)^i\beta_{ik}$	0	0
26	$\xi_{41}^{(k)}, \xi_{42}^{(k)}$	0	0	0	$(-1)^i q_+\alpha_{ik}$	$(-1)^i\alpha_{ik}$
27	${\xi'}_{41}^{(k)}, {\xi'}_{42}^{(k)}$	$(-1)^{i+1}\alpha_{jk}$	0	0	0	0
28	φ_1, φ_2	$(-1)^{i+1}$	$(-1)^i q_-$	$(-1)^{i+1}$	0	0
29	φ_3, φ_4	$(-1)^{i+1}$	$(-1)^{i+1} q_-$	$(-1)^i$	0	0
30	φ_5, φ_6	$(-1)^j$	0	0	$(-1)^i q_+$	$(-1)^i$
31	φ_7, φ_8	$(-1)^{j+1}$	0	0	$(-1)^i q_+$	$(-1)^i$
32	φ_9	0	q_-	-1	q_+	1
	Number of classes	$(q-1)(q-3)/4$	$(q-1)/2$	$(q-1)/2$	$(q-3)/2$	$(q-3)/2$

Table C′. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		16	17	18	19
		$c_1(i), c_1(i)'$ $(i \in T_2)$	$c_{21}(i), c_{21}(i)'$, $c_{22}(i), c_{22}(i)'$ $(i \in T_2)$	$c_3(i), c_3(i)'$ $(i \in T_1)$	$c_{41}(i), c_{41}(i)'$, $c_{42}(i), c_{42}(i)'$ $(i \in T_1)$
18	$\chi_8^{(k)}$	0	0	$q_+\alpha_{ik}$	α_{ik}
19	$\chi_9^{(k)}$	0	0	$q_+\alpha_{ik}$	α_{ik}
20	$\xi_1^{(k)}$	$q_- - \beta_{ik}$	$-\beta_{ik} - 1$	q_-	-1
21	${\xi'}_1^{(k)}$	$-q_- - q\beta_{ik}$	1	q_-	-1
22	$\xi_3^{(k)}$	q_+	1	$q_+ + \alpha_{ik}$	$\alpha_{ik} + 1$
23	${\xi'}_3^{(k)}$	$-q_+$	-1	$q_+ + q\alpha_{ik}$	1
24	$\xi_{21}^{(k)}, \xi_{22}^{(k)}$	$-q_+\beta_{ik}/2$	$\beta_{ik}b$	$(-1)^i q_-$	$(-1)^{i+1}$
25	${\xi'}_{21}^{(k)}, {\xi'}_{22}^{(k)}$	$(-1)^{i+1}q_- - q_-\beta_{ik}/2$	$(-1)^i - \beta_{ik}b$	0	0
26	$\xi_{41}^{(k)}, \xi_{42}^{(k)}$	0	0	$(-1)^i q_+ + q_+\alpha_{ik}/2$	$(-1)^i - \alpha_{ik}b$
27	${\xi'}_{41}^{(k)}, {\xi'}_{42}^{(k)}$	$(-1)^{i+1}q_+$	$(-1)^{i+1}$	$q_-\alpha_{ik}/2$	$\alpha_{ik}b$
28	φ_1, φ_2	$q_-/2 - (-1)^i$	$b^* - (-1)^i$	$q_-/2$	b
29	φ_3, φ_4	$-q_-/2 - (-1)^i q$	$-b^*$	$q_-/2$	b
30	φ_5, φ_6	$q_+/2$	$-b^*$	$q_+/2 + (-1)^i$	$-b + (-1)^i$
31	φ_7, φ_8	$-q_+/2$	b^*	$q_+/2 + (-1)^i q$	$-b$
32	φ_9	$(-1)^i q_+$	$(-1)^{i+1}$	$(-1)^i q_+$	$(-1)^i$
	Number of classes	$q-1$	$2(q-1)$	$q-3$	$2(q-3)$

Table D′. Character table of the group $\mathrm{Sp}_4(q)$, q is odd (continued)

		20	21	22	23
		d_1	d_{21}, d'_{21}, d_{22}, d'_{22}	d_{31}, d_{33}	d_{32}, d_{34}
18	$\chi_8^{(k)}$ $(k \in T_1)$	$(-1)^k {q_+}^2$	$(-1)^k q_+$	$(-1)^k$	$(-1)^k$
19	$\chi_9^{(k)}$ $(k \in T_1)$	$(-1)^k {q_+}^2$	$(-1)^k q_+$	$(-1)^k$	$(-1)^k$
20	$\xi_1^{(k)}$ $(k \in T_2)$	$q_- s(2,k)$	$q - s(2,k)$	$-s(2,k)$	$-s(2,k)$
21	${\xi'}_1^{(k)}$ $(k \in T_2)$	$qq_- s(2,k)$	$(-1)^{k+1} q$	0	0
22	$\xi_3^{(k)}$ $(k \in T_1)$	$q_+ s(2,k)$	$s(2,k) + q$	$s(2,k)$	$s(2,k)$
23	${\xi'}_3^{(k)}$ $(k \in T_1)$	$qq_+ s(2,k)$	$(-1)^k q$	0	0
24	$\xi_{21}^{(k)}, \xi_{22}^{(k)}$ $(k \in T_2)$	$(q^2-1)s(k,t)/2$	$(-1)^{k+1} q_+/2 - (-1)^t q_- b$	$s(k,t)b$	$(-1)^k b + (-1)^t b^*$
25	${\xi'}_{21}^{(k)}, {\xi'}_{22}^{(k)}$ $(k \in T_2)$	${q_-}^2 d(k,t)/2$	$(-1)^{k+1} q_-/2 - (-1)^t q_- b$	$-d(k,t)b$	$(-1)^{k+1} b + (-1)^t b^*$
26	$\xi_{41}^{(k)}, \xi_{42}^{(k)}$ $(k \in T_1)$	${q_+}^2 s(k,t)/2$	$(-1)^k q_+/2 - (-1)^t q_+ b$	$-s(k,t)b$	$(-1)^{k+1} b - (-1)^t b^*$
27	${\xi'}_{41}^{(k)}, {\xi'}_{42}^{(k)}$ $(k \in T_1)$	$-{q_-}^2 d(k,t)/2$	$(-1)^k q_-/2 - (-1)^t q_+ b$	$d(k,t)b$	$(-1)^k b - (-1)^t b^*$
28	φ_1, φ_2	$q_- d(2,t)/2$	$q_- d(2,t)/2 - q(-1)^t b$	$-d(2,t)/2$	$-d(2,t)/2$
29	φ_3, φ_4	$qq_- d(2,t)/2$	$q_-(-1)^t - (-1)^t b$	$s(1,t)(b^* - b)/2$	$s(2,t)(b - b^*)/2$
30	φ_5, φ_6	$q_+ s(2,t)/2$	$q_+ s(2,t)/2 + q(-1)^t b^*$	$s(2,t)/2$	$s(2,t)/2$
31	φ_7, φ_8	$qq_+ s(2,t)/2$	$q_+(-1)^t/2 + (-1)^t b^*$	$s(2,t)(b^* - b)/2$	$d(2,t)(b^* - b)/2$
32	φ_9	$(-1)^t(q^2+1)$	$(-1)^t$	$(-1)^t$	$(-1)^t$
	Number of classes	1	4	2	2

Define

$\mathcal{X} := \{\widetilde{1}, \widetilde{2}, \dots, \widetilde{32}\}$,

$\mathcal{Y}$ is the set of all rows $\widetilde{s}$ such that any character lying in $\widetilde{s}$ is not semiproportional to any character from any other row $\widetilde{t}$ (i.e., $\widetilde{s}$ is not semiproportional to $\widetilde{t} \neq \widetilde{s}$),

$\mathcal{Z} := \mathcal{X} \setminus \mathcal{Y}$.

We must show that $\mathcal{Z} = \varnothing$, i.e., that $\mathcal{Y} = \mathcal{X}$.

Since θ_0 $(= 1_G)$ has no zero values in contrast to other irreducible characters, we have $\widetilde{1} \in \mathcal{Y}$.

Since θ_{13} is the only character with the value 0 in the third column (i.e., on a_{21}), we have $\widetilde{10} \in \mathcal{Y}$.

Now, consider the rows from $\mathcal{X} \setminus \{\widetilde{1}, \widetilde{13}\}$ that have the value 0 in the fourth column. These are the rows $\widetilde{2}$, $\widetilde{4}$, $\widetilde{7}$, $\widetilde{9}$, $\widetilde{21}$, and $\widetilde{29}$. Consequently, they are not semiproportional to other rows. The rows $\widetilde{2}$ and $\widetilde{4}$ do not have equal roots with $\widetilde{7}$, $\widetilde{9}$, $\widetilde{21}$, and $\widetilde{29}$ on a_{22} and do not have equal roots with each other on $b_1(i)$. Hence, $\{\widetilde{2}, \widetilde{4}\} \subseteq \mathcal{Y}$.

Among the rows $\widetilde{7}$, $\widetilde{9}$, $\widetilde{21}$, and $\widetilde{29}$, only $\widetilde{7}$ has a nonzero value in the seventh column; hence, $\widetilde{7} \in \mathcal{Y}$.

Among the rows $\widetilde{9}$, $\widetilde{21}$, and $\widetilde{29}$, only $\widetilde{9}$ has a nonzero value on $\widehat{7}$; hence, $\widetilde{9} \in \mathcal{Y}$.

In addition, $\widetilde{21}$ and $\widetilde{29}$ are not semiproportional by statement (2) of Proposition 1.1 for $g_1 = a_1$ and $g_2 = a_{21}$ (here, $\widetilde{29}(a_1)/\widetilde{21}(a_1) = 1/2$, but the ratios $\varphi_3(a_{21})\widetilde{21}/(a_{21}) = -(q_-/2 + qb) = -(1 + \varepsilon q\sqrt{\varepsilon q})/2$ and $\varphi_4(a_{21})/\widetilde{21}(a_{21}) = -(q_-/2 + qb^*) = -(1 - \varepsilon q\sqrt{\varepsilon q})/2$ are different from $1/2$ and -2). Hence, $\{\widetilde{21}, \widetilde{29}\} \subseteq \mathcal{Y}$.

Thus,

$$\widetilde{1}, \widetilde{2}, \widetilde{4}, \widetilde{7}, \widetilde{9}, \widetilde{10}, \widetilde{21}, \widetilde{29} \in \mathcal{Y}, \tag{3.3}$$

and we will compare with each other only the remaining rows (they must contain the characters φ and ψ).

Among the rows not mentioned in (3.3), only $\widetilde{6}$, $\widetilde{8}$, $\widetilde{23}$, and $\widetilde{31}$ have zero in the fifth column. Among them, only $\widetilde{6}$ has a nonzero value on $b_1(i)$, and only $\widetilde{8}$ has a nonzero value on $b_2(i)$; i.e., $\{\widetilde{6}, \widetilde{8}\} \subseteq \mathcal{Y}$.

In addition, by statement (2) of Proposition 1.1, $\widetilde{23}$ and $\widetilde{31}$ are not semiproportional, which is seen from the comparison of their values on a_1 and a_{21} (here, $\widetilde{31}(a_1)/\widetilde{23}(a_1) = 1/2$, but the ratios $\varphi_7(a_{21})/\widetilde{23}(a_{21}) = q_+/2 + qb^* = (1 + \varepsilon q\sqrt{\varepsilon q})/2$ and $\varphi_8(a_{21})/\widetilde{23}(a_{21}) = q_+/2 + qb = (1 - \varepsilon q\sqrt{\varepsilon q})/2$ are different from $1/2$ and -2). Therefore, $\{\widetilde{23}, \widetilde{31}\} \subseteq \mathcal{Y}$.

Only $\widetilde{17}$, $\widetilde{19}$, and $\widetilde{32}$ have the first zero value in the sixth column. Among them, only $\widetilde{32}$ has a nonzero value in the ninth and tenth columns; consequently, $\widetilde{32} \in \mathcal{Y}$.

The degrees of the characters from $\widetilde{17}$ and $\widetilde{19}$ are different, but their values on a_{32} are equal in magnitude. By statement (3) of Proposition 1.1, these characters are not semiproportional; hence, $\widetilde{17}, \widetilde{19} \in \mathcal{Y}$.

Almost all of the rows not mentioned above have zero in the seventh column. Among them, only $\widetilde{3}$ has nonzero values on classes from B_2–B_4 (i.e., in $\widehat{8}$, $\widehat{9}$, and $\widehat{10}$). Consequently, $\widetilde{3} \in \mathcal{Y}$.

The characters from $\widetilde{5}$ and (possibly) some characters from $\widetilde{11}$ are the only characters that were not mentioned above and have a nonzero value in the seventh column. Among them, only $\widetilde{5}$ has a nonzero value on B_5 (i.e., in $\widehat{11}$), and, consequently, by (3.1), $\widetilde{5} \in \mathcal{Y}$.

Thus,

$$\widetilde{1}\text{–}\widetilde{10}, \widetilde{17}, \widetilde{19}, \widetilde{21}, \widetilde{23}, \widetilde{29}, \widetilde{31}, \widetilde{32} \in \mathcal{Y}. \tag{3.4}$$

Table 1

$\widetilde{s}$	$\widetilde{12}$	$\widetilde{13}$	$\widetilde{14}$	$\widetilde{15}$	$\widetilde{16}$	$\widetilde{18}$	$\widetilde{20}$	$\widetilde{22}$	$\widetilde{24}$	$\widetilde{25}$	$\widetilde{26}$	$\widetilde{27}$
$\widetilde{s}(1)$	Qq_-q_+	Qq_+^2	Qq_-^2	Qq_-q_+	Qq_-	Qq_+	Qq_-	Qq_-q_+	$Qq_-q_+/2$	$Qq_-^2/2$	$Qq_+^2/2$	$Qq_-q_+/2$

Table 2

$(\widetilde{m},\widetilde{n})$	$(\widetilde{12},\widetilde{24})$	$(\widetilde{12},\widetilde{27})$	$(\widetilde{13},\widetilde{26})$	$(\widetilde{14},\widetilde{25})$	$(\widetilde{15},\widetilde{24})$	$(\widetilde{15},\widetilde{27})$
$\widetilde{m}(1)/\widetilde{n}(1)$	2	2	2	2	2	2
$\widetilde{m}(a_{41})/\widetilde{n}(a_{41})$	$1/b^*, 1/b$	$1/b, 1/b^*$	$-1/b, -1/b^*$	$-1/b^*, -1/b$	$-1/b^*, -1/b$	$-1/b, -1/b^*$
$(\widetilde{m},\widetilde{n})$	$(\widetilde{24},\widetilde{20})$	$(\widetilde{24},\widetilde{22})$	$(\widetilde{25},\widetilde{20})$	$(\widetilde{26},\widetilde{22})$	$(\widetilde{27},\widetilde{20})$	$(\widetilde{27},\widetilde{22})$
$\widetilde{m}(1)/\widetilde{n}(1)$	$q_+/2$	$q_-/2$	$q_-/2$	$q_+/2$	$q_+/2$	$q_-/2$
$\widetilde{m}(a_{41})/\widetilde{n}(a_{41})$	$-b^*,\ -b$	$b^*,\ b$	$b^*,\ b$	$-b,\ -b^*$	$-b,\ -b^*$	$b\ , b^*$

It remains to consider the rows $\widetilde{11}$–$\widetilde{16}$, $\widetilde{18}$, $\widetilde{20}$, $\widetilde{22}$, $\widetilde{24}$–$\widetilde{28}$, and $\widetilde{30}$. Among them, the degrees of characters are not multiples of q^2+1 only in $\widetilde{11}$, and, consequently, by statement (3.1), $\widetilde{11}\in\mathcal{Y}$.

In addition, the row $\widetilde{28}$ cannot have equal roots with any of the rows $\widetilde{12}$–$\widetilde{16}$, $\widetilde{18}$, $\widetilde{22}$, and $\widetilde{24}$–$\widetilde{30}$ (see the values on B_5, B_6, and B_7), and $\widetilde{28}$ is not semiproportional to $\widetilde{20}$ by statement (2) of Proposition 1.1 (consider the values on a_1 and a_{41}). Consequently, $\widetilde{28}\in\mathcal{Y}$.

Similarly, $\widetilde{30}$ cannot have equal roots with any of the rows $\widetilde{11}$–$\widetilde{16}$, $\widetilde{18}$, $\widetilde{20}$, and $\widetilde{24}$–$\widetilde{28}$ (see the values on B_5, B_6, and B_9), and $\widetilde{30}$ is not semiproportional to $\widetilde{22}$ by statement (2) of Proposition 1.1 (consider the values on a_1 and a_{41}). Consequently, $\widetilde{30}\in\mathcal{Y}$.

Thus,

$$\widetilde{1}\text{–}\widetilde{11}, \widetilde{17}, \widetilde{19}, \widetilde{21}, \widetilde{23}, \widetilde{28}\text{–}\widetilde{32}\in\mathcal{Y} \tag{3.5}$$

and $\mathcal{X}\setminus\mathcal{Y}\subseteq\{\widetilde{12}\text{–}\widetilde{16}, \widetilde{18}, \widetilde{20}, \widetilde{22}, \widetilde{24}\text{–}\widetilde{27}\}$. The rows $\widetilde{12}$–$\widetilde{16}$, $\widetilde{18}$, $\widetilde{20}$, and $\widetilde{22}$ have the values ±1 on a_{41}, and, consequently, by (3.2),

$$\text{the rows } \widetilde{12}\text{–}\widetilde{16},\ \widetilde{18},\ \widetilde{20}, \text{ and } \widetilde{22} \text{ are pairwise not semiproportional.} \tag{3.6}$$

In addition, the degrees of the characters from different rows among $\widetilde{24}$, $\widetilde{25}$, $\widetilde{26}$, and $\widetilde{27}$ do not divide each other pairwise (see Table 1). Consequently, by (3.1),

$$\text{the rows } \widetilde{24},\ \widetilde{25},\ \widetilde{26}, \text{ and } \widetilde{27} \text{ are pairwise not semiproportional.} \tag{3.7}$$

Let us write the degrees $\widetilde{s}(1)$ of characters from the rows $\widetilde{s}$ in (3.6) and (3.7) (see Table 1; here, $Q := q^2+1$).

We can assume that $\varphi\in\widetilde{m}$ with $m\in\{12,13,14,15,15,18,20,22\}$ and $\psi\in\widetilde{n}$ with $n\in\{24,25,26,27\}$. According to (3.1), up to the order, only the following pairs $(\widetilde{m},\widetilde{n})$ (for $m\neq n$) are possible (see Table 2).

It is seen from Table 2 that the number $\widetilde{m}(a_{41})/\widetilde{n}(a_{41})$ in each of these 12 cases is different both from $\widetilde{m}(1)/\widetilde{n}(1)$ and from $-\widetilde{n}(1)/\widetilde{m}(1)$; however, this contradicts statement (2) of Proposition 1.1.

Thus, $\{\varphi,\psi\}\subseteq\widetilde{i}$, where $i\in\{1,\dots,32\}$; i.e., $m=n$ and, in particular, $\varphi(1)=\psi(1)$, which means the validity of statement (1) of the theorem.

To prove statement (2), it remains to exclude some i in the previous sentence. More exactly, we must show that $\{m,n\} \subseteq \widetilde{i}$, where $11 \le i \le 27$. This is easily seen from the values of the characters on $c_{41}(1)$ (if, for example, $\{\varphi,\psi\} \subseteq \widetilde{2}$, then $\varphi(c_{41}(1))/\psi(c_{41}(1)) = b/b^* \notin \{1,-1\}$).

The theorem is proved.

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